SEARCHING FOR THREE-BODY PERIODIC ORBITS AND COMPUTING THEM WITH HIGH ACCURACY

Ivan Hristov
Faculty of Mathematics and Informatics,
Sofia University "St. Kliment Ohridski"

Seminar at
Faculty of mathematics and Informatics
Sofia University "St. Kliment Ohridski"
January 21-th, 2025

What exactly is the three-body problem? What means that it is "unsolvable"?

Problem formulation: Three bodies move in space under their mutual gravitational attraction. Given their initial conditions (initial positions and velocities), determine their subsequent motion.

In 1887 Ernst Heinrich Bruns (professor of astronomy in Leipzig) showed that there are 18 degrees of freedom, but only 10 integrals of motion in the dynamics of three bodies. From this result follows that there is no general closed-form solution to the three-body problem. In other words, it does not have a general solution that can be expressed in terms of a finite number of standard mathematical operations.

Moreover, as Henri Poincaré shows in 1890, the motion is mathematically chaotic, particularly it has a sensitive dependence on the initial conditions.

Henri Poincaré (1854-1912)

"... what makes these (periodic) solutions so precious to us, is that they are, so to say, the only opening through which we can try to penetrate in a place which, up to now, was supposed to be inaccessible."

Why are periodic orbits important?

A) The periodic orbits existence can be rigourously proved. Different methods from differential equations theory can be implemented, like fixed point theorem, power or Fourier series expansion, et cetera.

B) The knowledge of one period gives the full knowledge of the solution, which simplifies the qualitative analysis of the solution.

C) The periodic orbits can be computed for all time with any given accuracy.

D) The unstable periodic orbits provide critical information in chaotic regions.

Why are periodic orbits important?

Definition: Call a motion *bounded* if the distance between the three bodies remains bounded as a function of time, and *unbounded* otherwise.

E) According to Kolmogorov-Arnold-Moser (KAM)-theory stable periodic orbits are surrounded by sets of orbits with bounded motions.

F) The most interesting feature of periodic orbits remains the still unproved **Poincaré "other conjecture"**:

The periodic solutions of the three-body problem are dense in the set of bounded solutions.

If true this conjecture would show that the periodic solutions are a very efficient mean of "exploration".

Another open problem: Is it true that arbitrarily close to any bounded solution lies an unbounded solution?

Three-body problem and chaos

The three-body system is habitually described as chaotic. However, there are plenty of stable periodic equal-mass three-body orbits, which contradicts the requirement that a chaotic system has only unstable periodic orbits. Manifestly, only a subset of the full equal-mass three-body problem can be chaotic.

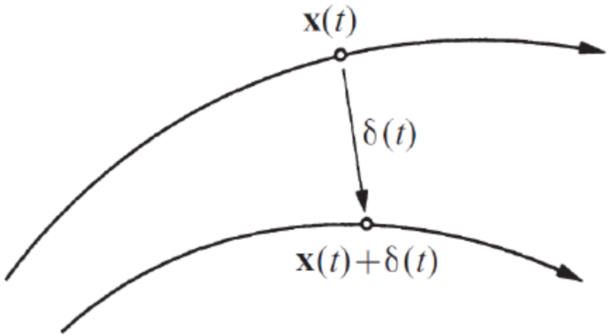
No universally accepted mathematical definition of chaos exists. A simple and widely used definition given by Robert L. Devaney, says that to classify a dynamical system as chaotic, it must have these properties:

1) It must be sensitive to initial conditions.

2) It must have dense periodic orbits.

3) It must have topological mixing.

Sensitive dependence on initial conditions



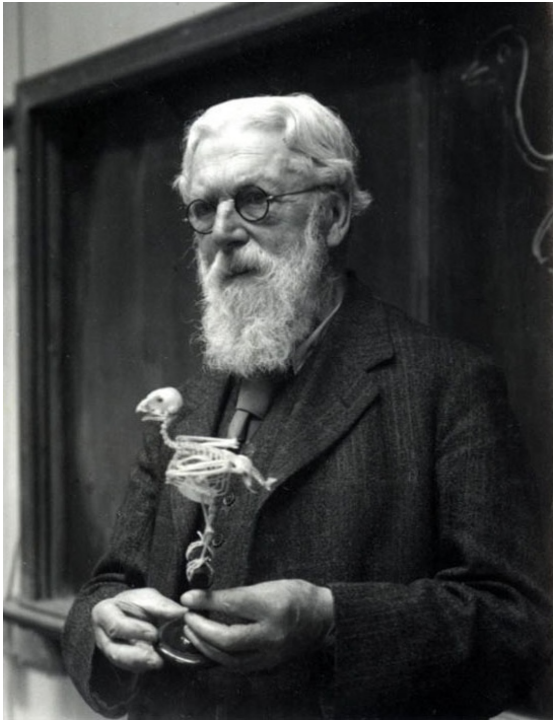
$$\delta(t) \approx \delta(0)e^{\lambda t}$$

$\lambda > 0$ is the Lyapunov exponent
Predictability horizont (Lyapunov time) T is defined by

$$T = \frac{1}{\lambda}\ln(\frac{tol}{\varepsilon})$$

where tol is our tolerance and ε is the round-off unit (precision)

"...Numerical precision is the very soul of science, and its attainment affords the best, perhaps the only criterion of the truth of theories and the correctness of experiments."



Sir D'Arcy Wentworth Thompson (1860-1948), On Growth and Form

What we generally need to obtain a reliable long-term solution when the solution is sensitive?

We need to combine:

1. A multiple-precision floating point arithmetic

with

2. A class of numerical methods allowing arbitrary high order of accuracy.

In our work we use:

1. GMP library (The GNU Multiple Precision Arithmetic Library).

2. Taylor series method (TSM) as an ODE-solver.

PRL **110,** 114301 (2013) PHYSICAL REVIEW LETTERS week ending 15 MARCH 2013

Three Classes of Newtonian Three-Body Planar Periodic Orbits

Milovan Šuvakov* and V. Dmitrašinović

Institute of Physics Belgrade, University of Belgrade, Pregrevica 118, 11080 Belgrade, Serbia

(Received 2 November 2012; published 14 March 2013)

We present the results of a numerical search for periodic orbits of three equal masses moving in a plane under the influence of Newtonian gravity, with zero angular momentum. A topological method is used to classify periodic three-body orbits into families, which fall into four classes, with all three previously known families belonging to one class. The classes are defined by the orbits' geometric and algebraic symmetries. In each class we present a few orbits' initial conditions, 15 in all; 13 of these correspond to distinct orbits.

DOI: 10.1103/PhysRevLett.110.114301 PACS numbers: 45.50.Jf, 05.45.−a, 95.10.Ce

SCIENCE CHINA
Physics, Mechanics & Astronomy

CrossMark click for updates

• Article •

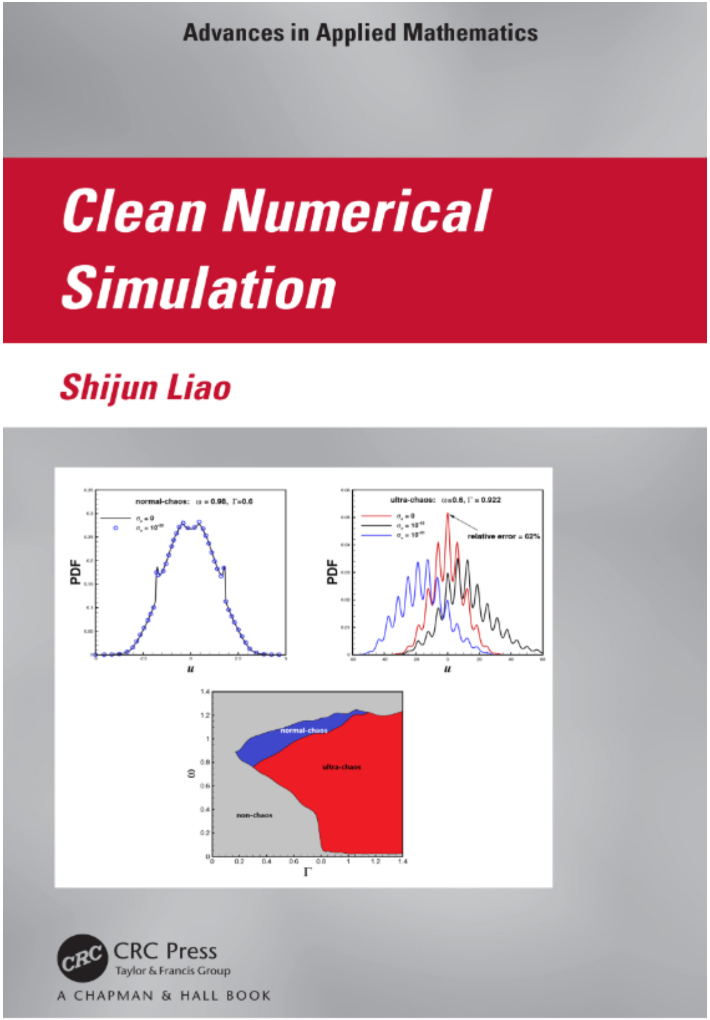
December 2017 Vol. 60 No. 12: 129511
doi: 10.1007/s11433-017-9078-5

More than six hundred new families of Newtonian periodic planar collisionless three-body orbits

XiaoMing Li[1], and ShiJun Liao[1,2*]

[1] *School of Naval Architecture, Ocean and Civil Engineering, Shanghai Jiaotong University, Shanghai 200240, China;*
[2] *MoE Key Laboratory in Scientific and Engineering Computing, Shanghai 200240, China*

Received July 4, 2017; accepted July 11, 2017; published online September 11, 2017


Advances in Applied Mathematics
Clean Numerical Simulation
Shijun Liao
normal-chaos
ultra-chaos
non-chaos
relative error = 62%
PDF
CRC Press
Taylor & Francis Group
A CHAPMAN & HALL BOOK

arXiv > nlin > arXiv:2111.10839

Search...

Help | Adva

Nonlinear Sciences > Chaotic Dynamics

[Submitted on 21 Nov 2021]

Newton's method for computing periodic orbits of the planar three-body problem

I. Hristov, R. Hristova, I. Puzynin, T. Puzynina, Z. Sharipov, Z. Tukhliev

In this paper we present in detail Newton's method and its modification, based on the Continuous analog of Newton's method for computing periodic orbits of the planar three-body problem. The linear system at each step of the method is formed by solving a system of ODEs with the multiple precision Taylor series method. We consider zero angular momentum symmetric initial configuration with parallel velocities, bodies with equal masses and relatively short periods. Taking candidates for the correction method with greater return proximity as usual and correcting with the modified Newton's method, allows us to find some new topological families that are not included in the database in [SCIENCE CHINA Physics, Mechanics & Astronomy 60.12 (2017)]

Comments: 11 pages, 1 figure, 2 tables

Subjects: **Chaotic Dynamics (nlin.CD)**; Numerical Analysis (math.NA); Computational Physics (physics.comp-ph)

Recently published papers

Hristov, I., Hristova, R., Dmitrašinović, V., Tanikawa, K.
"Three-body periodic collisionless equal-mass free-fall orbits revisited"
Celestial Mechanics and Dynamical Astronomy, 136(1), 7, 2024

Hristov, I., Hristova, R., Dmitrašinović, V., Tanikawa, K.
"Instability of three-body periodic collisionless equal-mass free-fall orbits"
In Journal of Physics: Conference Series (Vol. 2910, No. 1, p. 012030). IOP Publishing, 2024

Hristov, I., Hristova, R. *"An efficient approach for searching three-body periodic orbits passing through Eulerian configuration"*
Astronomy and Computing, 49, 100880, 2024

Hristov, I., Hristova, R., Puzynin, I., Puzynina, T., Sharipov,Z., Tukhliev,Z.
"Searching for New Nontrivial Choreographies for the Planar Three-Body Problem" Physics of Particles and Nuclei, 55(3), 495-497, 2024

Differential equations

The differential equations for the three-body problem are derived from Newton's second law and Newton's law of gravity:

$$m_i \ddot{r}_i = \sum_{j=1, j\neq i}^{3} G m_i m_j \frac{(r_j - r_i)}{\|r_i - r_j\|^3}, i = 1, 2, 3.$$

We consider normalization $G = m_1 = m_2 = m_3 = 1$ and planar motion. We solve the system numerically in the following first order form:

$$\dot{x}_i = vx_i,\ \dot{y}_i = vy_i$$

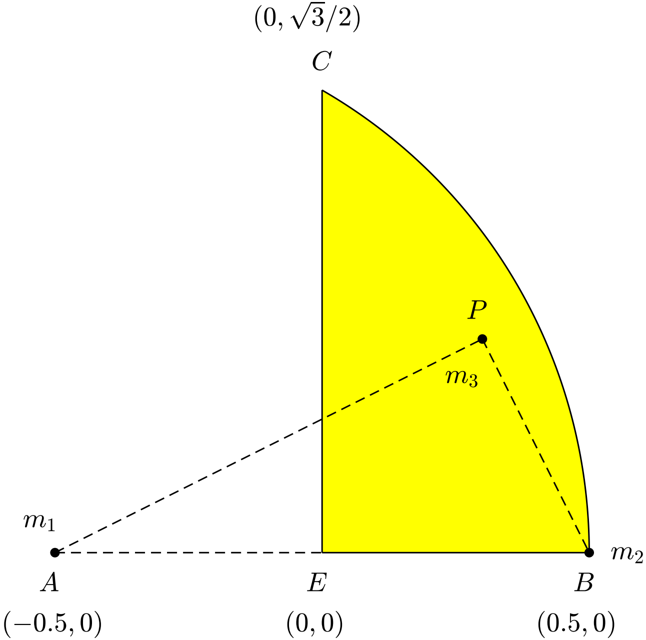
$$\dot{vx}_i = \sum_{j=1, j\neq i}^{3} \frac{(x_j - x_i)}{\|r_i - r_j\|^3},\ \dot{vy}_i = \sum_{j=1, j\neq i}^{3} \frac{(y_j - y_i)}{\|r_i - r_j\|^3},\ i = 1, 2, 3$$

So we have a vector of 12 unknown functions:

$$X(t) = (x_1, y_1, x_2, y_2, x_3, y_3, vx_1, vy_1, vx_2, vy_2, vx_3, vy_3)^\top$$

The model treats the bodies as point masses.

Free-fall initial conditions. Agekyan-Anosova's domain



$$(x_1(0), y_1(0)) = (-0.5, 0),\ \ (x_2(0), y_2(0)) = (0.5, 0)$$
$$(x_3(0), y_3(0)) = (p_x, p_y)$$
$$(vx_1(0), vy_1(0)) = (vx_2(0), vy_2(0)) = (vx_3(0), vy_3(0)) = (0, 0)$$

We have two parameters p_x, p_y - the coordinates of the point P in the Agekyan-Anosova's domain.

Back
Close

Euler initial conditions (Euler half-twist conditions)

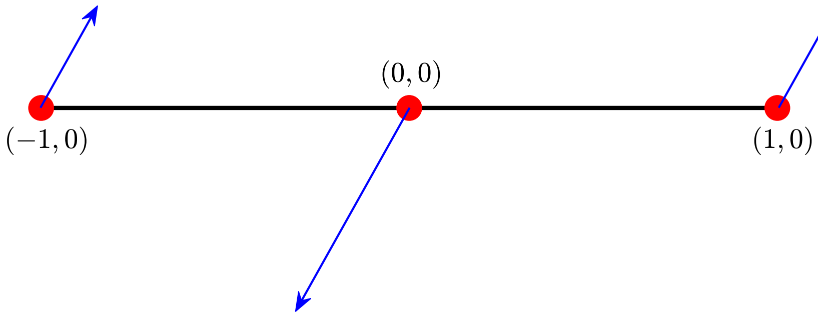
Euler initial conditions are the following symmetrical collinear with parallel velocities conditions with two parameters $v_x > 0$, $v_y > 0$:

$$(x_1(0), y_1(0)) = (-1, 0), (x_2(0), y_2(0)) = (1, 0)$$

$$(x_3(0), y_3(0)) = (0, 0)$$

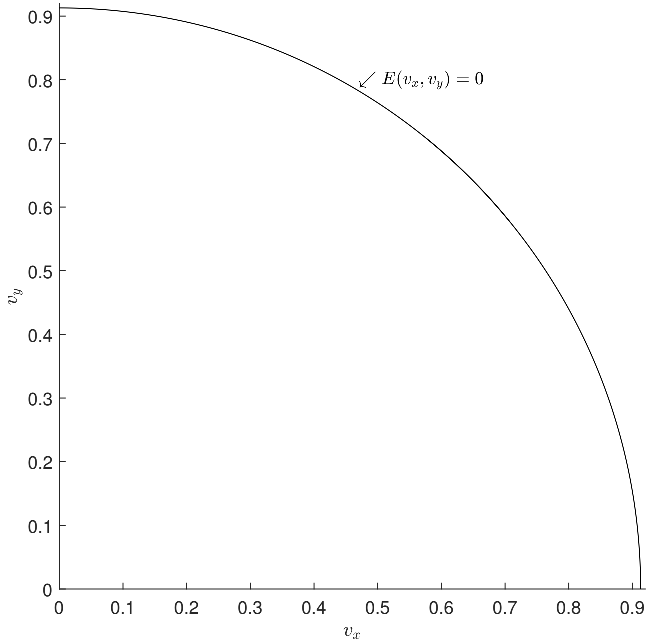
$$(vx_1(0), vy_1(0)) = (vx_2(0), vy_2(0)) = (v_x, v_y)$$

$$(vx_3(0), vy_3(0)) = -2(vx_1(0), vy_1(0)) = (-2v_x, -2v_y)$$



For initial positions $x_i(0), y_i(0)$ fixed as above, the two-dimensional space of velocities is precisely the space of all velocities for which the linear and angular momentum are zero and for which the moment of inertia $I = {x_1}^2 + {y_1}^2 + {x_2}^2 + {y_2}^2 + {x_3}^2 + {y_3}^2$ has an extremum at $t = 0$.

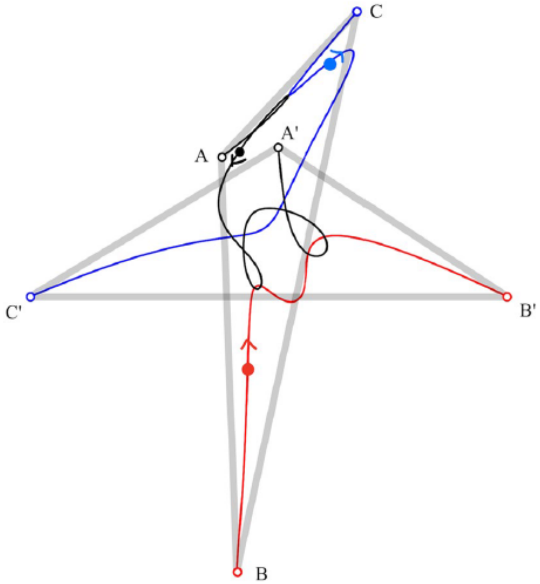
Euler initial conditions: The 2D search domain



As only negative energies (only bounded motions) have to be considered, the 2D search domain is actually those bounded by $\boldsymbol{v_x = 0}$ and $\boldsymbol{v_y = 0}$ axis and $\boldsymbol{E = 0}$ curve. $\boldsymbol{E = -2.5 + 3({v_x}^2 + {v_y}^2) = 0}$.

The new searching approach for free-fall periodic orbits

The free-fall periodic orbits shuttle back and forth between two stop triangles. Instead of looking for orbits that satisfy the standard periodicity condition, we could look for orbits that stop at some later time - an approximation of $\boldsymbol{T/2}$.



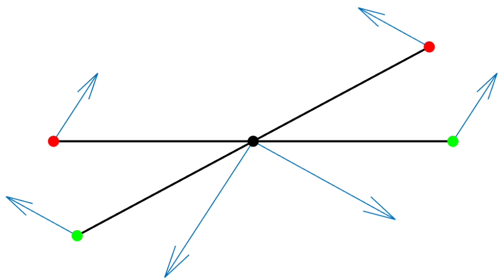
Picture from *R. Montgomery "Dropping bodies" The Mathematical Intelligencer 45.2 (2023): 168-174*

The new searching approach for Eulerian periodic orbits

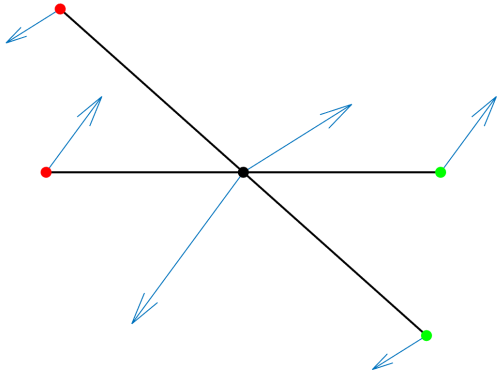
The orbits in the basic Šuvakov and Dmitrašinović paper are divided into two types: *"(I) those with reflection symmetries about two orthogonal axes on the shape sphere - the equator and the zeroth meridian and (II) those with a central reflection symmetry about a single point - the intersection of the equator and zeroth meridian".*

The main observation is that the bodies pass again through Euler configuration at the half period!

type (I)



type (II)



If this property can be proved, than instead of looking for orbits that satisfy the standard periodicity condition, we could look for orbits that pass again through Euler configuration at some later time.

The "half period" property was proved by prof. Richard Montgomery

If a three-body solution $\boldsymbol{r}(\boldsymbol{t})$ *has Eulerian half-twist initial conditions at time* $\boldsymbol{t} = \boldsymbol{0}$ *and is periodic of period* $\boldsymbol{T}$ *then it has Eulerian half-twist initial conditions at the half period time* $\boldsymbol{t} = \boldsymbol{T}/\boldsymbol{2}$.

Lyapunov exponents' argument, which tries to explain the observed high efficiency of the "half period" approach

Let us assume unstable periodic orbits and that for small separation $d(t) = \|p(t) - \widetilde{p}(t)\|_2$ between adjacent trajectories the exponential law of divergence is satisfied:

$$d(t) \approx d(0)e^{\lambda t}, \quad \lambda - \text{the Lyapunov exponent}$$

Here $p(t)$ is a given unstable periodic orbit and $\widetilde{p}(t)$ is an approximation of it. What is the effect of the integration time division by two, i.e. the effect of solving the Euler equation at $t = T/2$ instead of solving the periodic equation at $t = T$? The benefit is not simply reducing the computational time by two, but much more. We have a "square root effect" on the distance between adjacent trajectories, meaning that:

$$d(T/2) \approx d(0)\sqrt{e^{\lambda T}}, \quad T - \text{the period}$$

Let us take for example $d(0) \approx 10^{-4}$ and take $e^{\lambda T} \approx 10^{8}$. Then $d(T/2) \approx 1$, but $d(T) \approx 10^{4}$.

Return proximity function for the standard periodicity condition (equation)

Let us denote the solution with Euler initial conditions with $\boldsymbol{X}(v_x, v_y, t)$. If we denote the periods of the orbits with $\boldsymbol{T}$, then the goal is to find triplets (v_x, v_y, T) for which the periodicity condition (equation):

$$\boldsymbol{X}(v_x, v_y, T) = \boldsymbol{X}(v_x, v_y, 0)$$

is satisfied. The function

$$R(t) = \|\boldsymbol{X}(v_x, v_y, t) - \boldsymbol{X}(v_x, v_y, 0)\|_2, \quad t > 0$$

is called the return proximity function. The triplet (v_x, v_y, T), $T > 0$ corresponds to a periodic solution only if $R(T) = 0$. For approximate solution with approximations (v_x, v_y, T), $R(T)$ is the measure of how close to a periodic solution we are. $R(T)$ is the residual in which terms we define the convergence of the Newton's method for the standard periodicity condition.

Proximity function for the “half period” Eulerian condition

If we introduce the vectors $\boldsymbol{X}_1(\boldsymbol{t}) = (\boldsymbol{x_1}, \boldsymbol{y_1}, \boldsymbol{vx_1}, \boldsymbol{vy_1})^\top$, $\boldsymbol{X}_2(\boldsymbol{t}) = (-\boldsymbol{x_2}, -\boldsymbol{y_2}, \boldsymbol{vx_2}, \boldsymbol{vy_2})^\top$, the “half period” Eulerian condition becomes:

$$\boldsymbol{X}_1(\boldsymbol{v_x}, \boldsymbol{v_y}, \boldsymbol{T}/2) = \boldsymbol{X}_2(\boldsymbol{v_x}, \boldsymbol{v_y}, \boldsymbol{T}/2)$$

The proximity function for Eulerian condition is defined as:

$$\boldsymbol{R_e}(\boldsymbol{t}) = \|\boldsymbol{X}_1(\boldsymbol{v_x}, \boldsymbol{v_y}, \boldsymbol{t}) - \boldsymbol{X}_2(\boldsymbol{v_x}, \boldsymbol{v_y}, \boldsymbol{t})\|_2, \quad \boldsymbol{t} > 0$$

The triplet $(\boldsymbol{v_x}, \boldsymbol{v_y}, \overline{\boldsymbol{T}})$, $\overline{\boldsymbol{T}} > 0$ corresponds to a solution of the Eulerian condition (equation) at $\overline{\boldsymbol{T}}$ only if $\boldsymbol{R_e}(\overline{\boldsymbol{T}}) = 0$. $\overline{\boldsymbol{T}}$ is an approximation of $\boldsymbol{T}/2$.

Stages of the numerical search

Stage I: **Scanning stage:** Candidates for correction (initial approximations) are computed by scanning the initial condition domain with the grid-search algorithm. At each grid point (v_x, v_y) we simulate the three-body ODE system with the Euler initial condition up to some pre-defined value of time $T_0/2$. At each grid point (v_x, v_y) we compute the time $\overline{T}$ at which the minimum:

$$R_e(\overline{T}) = \min_{1<t<T_0/2} R_e(t)$$

is obtained. Candidates are those triplets $(v_x, v_y, \overline{T})$ with small values of the proximity function $R_e(\overline{T})$.

Stage II: **Capturing stage:** The periodic solutions are captured with modified Newton's method starting with initial approximations obtained in stage I. The convergence is in terms of the proximity function $R_e(\overline{T})$.

Stage III: **Verification stage:** The solutions are computed with many correct digits and the convergence of Newton's method is checked. The convergence is in terms of the return proximity function $R(T)$.

The linear system at each Newton's iteration at stage I,II (the linear system with respect to the Eulerian condition)

Let $\boldsymbol{\overline{T}}$ be an approximation of $\boldsymbol{T/2}$ and the triplet $(\boldsymbol{v_x}, \boldsymbol{v_y}, \boldsymbol{\overline{T}})$ be an approximation of the "half period" Eulerian condition (equation) at $\boldsymbol{T/2}$, i.e. $\boldsymbol{X_1(v_x, v_y, \overline{T}) \approx X_2(v_x, v_y, \overline{T})}$.

Then the Eulerian equation with corrections $\boldsymbol{\Delta v_x}, \boldsymbol{\Delta v_y}, \boldsymbol{\Delta \overline{T}}$ is:

$$\boldsymbol{X_1(v_x + \Delta v_x, v_y + \Delta v_y, \overline{T} + \Delta\overline{T}) = X_2(v_x + \Delta v_x, v_y + \Delta v_y, \overline{T} + \Delta\overline{T})}$$

Expanding this equation in a multivariable linear approximation gives:

$$\begin{pmatrix} x_1(\overline{T}) \\ y_1(\overline{T}) \\ vx_1(\overline{T}) \\ vy_1(\overline{T}) \end{pmatrix} + \begin{pmatrix} \frac{\partial x_1}{\partial v_x}(\overline{T}) & \frac{\partial x_1}{\partial v_y}(\overline{T}) & \dot{x}_1(\overline{T}) \\ \frac{\partial y_1}{\partial v_x}(\overline{T}) & \frac{\partial y_1}{\partial v_y}(\overline{T}) & \dot{y}_1(\overline{T}) \\ \frac{\partial vx_1}{\partial v_x}(\overline{T}) & \frac{\partial vx_1}{\partial v_y}(\overline{T}) & \dot{vx}_1(\overline{T}) \\ \frac{\partial vy_1}{\partial v_x}(\overline{T}) & \frac{\partial vy_1}{\partial v_y}(\overline{T}) & \dot{vy}_1(\overline{T}) \end{pmatrix} \begin{pmatrix} \Delta v_x \\ \Delta v_y \\ \Delta \overline{T} \end{pmatrix} =$$

$$\begin{pmatrix} -x_2(\overline{T}) \\ -y_2(\overline{T}) \\ vx_2(\overline{T}) \\ vy_2(\overline{T}) \end{pmatrix} + \begin{pmatrix} -\frac{\partial x_2}{\partial v_x}(\overline{T}) & -\frac{\partial x_2}{\partial v_y}(\overline{T}) & -\dot{x}_2(\overline{T}) \\ -\frac{\partial y_2}{\partial v_x}(\overline{T}) & -\frac{\partial y_2}{\partial v_y}(\overline{T}) & -\dot{y}_2(\overline{T}) \\ \frac{\partial vx_2}{\partial v_x}(\overline{T}) & \frac{\partial vx_2}{\partial v_y}(\overline{T}) & \dot{vx}_2(\overline{T}) \\ \frac{\partial vy_2}{\partial v_x}(\overline{T}) & \frac{\partial vy_2}{\partial v_y}(\overline{T}) & \dot{vy}_2(\overline{T}) \end{pmatrix} \begin{pmatrix} \Delta v_x \\ \Delta v_y \\ \Delta \overline{T} \end{pmatrix}$$

The linear system at each Newton's iteration at stage I,II (the linear system with respect to the Eulerian condition)

Finally, we obtain the following linear system with a $\mathbf{4 \times 3}$ matrix with respect to $(\boldsymbol{\Delta v_x}, \boldsymbol{\Delta v_y}, \boldsymbol{\Delta \overline{T}})^\top$, that have to be solved at each Newton's iteration.

$$\begin{pmatrix} \frac{\partial x_1}{\partial v_x}(\overline{T}) + \frac{\partial x_2}{\partial v_x}(\overline{T}) & \frac{\partial x_1}{\partial v_y}(\overline{T}) + \frac{\partial x_2}{\partial v_y}(\overline{T}) & \dot{x}_1(\overline{T}) + \dot{x}_2(\overline{T}) \\ \frac{\partial y_1}{\partial v_x}(\overline{T}) + \frac{\partial y_2}{\partial v_x}(\overline{T}) & \frac{\partial y_1}{\partial v_y}(\overline{T}) + \frac{\partial y_2}{\partial v_y}(\overline{T}) & \dot{y}_1(\overline{T}) + \dot{y}_2(\overline{T}) \\ \frac{\partial vx_1}{\partial v_x}(\overline{T}) - \frac{\partial vx_2}{\partial v_x}(\overline{T}) & \frac{\partial vx_1}{\partial v_y}(\overline{T}) - \frac{\partial vx_2}{\partial v_y}(\overline{T}) & \dot{vx}_1(\overline{T}) - \dot{vx}_2(\overline{T}) \\ \frac{\partial vy_1}{\partial v_x}(\overline{T}) - \frac{\partial vy_2}{\partial v_x}(\overline{T}) & \frac{\partial vy_1}{\partial v_y}(\overline{T}) - \frac{\partial vy_2}{\partial v_y}(\overline{T}) & \dot{vy}_1(\overline{T}) - \dot{vy}_2(\overline{T}) \end{pmatrix} \begin{pmatrix} \Delta v_x \\ \Delta v_y \\ \Delta \overline{T} \end{pmatrix} = \begin{pmatrix} -x_1(\overline{T}) - x_2(\overline{T}) \\ -y_1(\overline{T}) - y_2(\overline{T}) \\ vx_2(\overline{T}) - vx_1(\overline{T}) \\ vy_2(\overline{T} - vy_1(\overline{T}) \end{pmatrix}$$

The linear systems are solved in least square sense by QR-decomposition based on Householder reflections!

The linear system at each Newton's iteration at stage III

Let $(\boldsymbol{v_x}, \boldsymbol{v_v}, \boldsymbol{T})$ be an approximation for a periodic orbit, i.e. $\boldsymbol{X}(\boldsymbol{v_x}, \boldsymbol{v_y}, \boldsymbol{T}) \approx \boldsymbol{X}(\boldsymbol{v_x}, \boldsymbol{v_y}, \boldsymbol{0})$. These approximations are improved with corrections $\boldsymbol{\Delta v_x}, \boldsymbol{\Delta v_y}, \boldsymbol{\Delta T}$ by expanding the periodicity condition:

$$\boldsymbol{X}(\boldsymbol{v_x} + \boldsymbol{\Delta v_x}, \boldsymbol{v_y} + \boldsymbol{\Delta v_y}, \boldsymbol{T} + \boldsymbol{\Delta T}) = \boldsymbol{X}(\boldsymbol{v_x} + \boldsymbol{\Delta v_x}, \boldsymbol{v_y} + \boldsymbol{\Delta v_y}, \boldsymbol{0})$$

in a multivariable linear approximation:

$$\begin{pmatrix} x_1(T) \\ y_1(T) \\ x_2(T) \\ y_2(T) \\ x_3(T) \\ y_3(T) \\ vx_1(T) \\ vy_1(T) \\ vx_2(T) \\ vy_2(T) \\ vx_3(T) \\ vy_3(T) \end{pmatrix} + \begin{pmatrix} \frac{\partial x_1}{\partial v_x}(T) & \frac{\partial x_1}{\partial v_y}(T) & \dot{x}_1(T) \\ \frac{\partial y_1}{\partial v_x}(T) & \frac{\partial y_1}{\partial v_y}(T) & \dot{y}_1(T) \\ \frac{\partial x_2}{\partial v_x}(T) & \frac{\partial x_2}{\partial v_y}(T) & \dot{x}_2(T) \\ \frac{\partial y_2}{\partial v_x}(T) & \frac{\partial y_2}{\partial v_y}(T) & \dot{y}_2(T) \\ \frac{\partial x_3}{\partial v_x}(T) & \frac{\partial x_3}{\partial v_y}(T) & \dot{x}_3(T) \\ \frac{\partial y_3}{\partial v_x}(T) & \frac{\partial y_3}{\partial v_y}(T) & \dot{y}_3(T) \\ \frac{\partial vx_1}{\partial v_x}(T) & \frac{\partial vx_1}{\partial v_y}(T) & \dot{vx}_1(T) \\ \frac{\partial vy_1}{\partial v_x}(T) & \frac{\partial vy_1}{\partial v_y}(T) & \dot{vy}_1(T) \\ \frac{\partial vx_2}{\partial v_x}(T) & \frac{\partial vx_2}{\partial v_y}(T) & \dot{vx}_2(T) \\ \frac{\partial vy_2}{\partial v_x}(T) & \frac{\partial vy_2}{\partial v_y}(T) & \dot{vy}_2(T) \\ \frac{\partial vx_3}{\partial v_x}(T) & \frac{\partial vx_3}{\partial v_y}(T) & \dot{vx}_3(T) \\ \frac{\partial vy_3}{\partial v_x}(T) & \frac{\partial vy_3}{\partial v_y}(T) & \dot{vy}_3(T) \end{pmatrix} \begin{pmatrix} \Delta v_x \\ \Delta v_y \\ \Delta T \end{pmatrix} = \begin{pmatrix} x_1(0) \\ y_1(0) \\ x_2(0) \\ y_2(0) \\ x_3(0) \\ y_3(0) \\ vx_1(0) + \Delta v_x \\ vy_1(0) + \Delta v_y \\ vx_2(0) + \Delta v_x \\ vy_2(0) + \Delta v_y \\ vx_3(0) - 2\Delta v_x \\ vy_3(0) - 2\Delta v_y \end{pmatrix}$$

Back
Close

Modified Newton's method (Igor Viktorovich Puzynin)

Let the triplet $(v_x, v_y, \overline{T})$ be an approximate solution of the "half period" Eulerian condition (equation). We correct and obtain the next approximation with the classic Newton's method this way:

$$v_x := v_x + \Delta v_x,\ \ v_y := v_y + \Delta v_y,\ \ \overline{T} := \overline{T} + \Delta\overline{T}$$

At stage II of the numerical search we use a modification of Newton's method based on the continuous analog of Newton's method. We introduce the parameter p_k: $0 < p_k \leq 1$, where k is the number of the iteration. Now we correct this way:

$$v_x := v_x + p_k\Delta v_x,\ \ v_y := v_y + p_k\Delta v_y,\ \ \overline{T} := \overline{T} + p_k\Delta\overline{T}$$

Let R_k be the value of the proximity function $R_e(\overline{T})$ (the residual) at the k-th iteration. With a given p_0 the next $p_k, k = 1, 2, \ldots$ is computed with the following adaptive algorithm:

$$p_k = \begin{cases} \min(1,\ p_{k-1}R_{k-1}/R_k), & R_k \leq R_{k-1}, \\ \max(p_0,\ p_{k-1}R_{k-1}/R_k), & R_k > R_{k-1}, \end{cases}$$

Computing the coefficients of the linear system

To compute the coefficients in the $\mathbf{12 \times 3}$ matrices or $\mathbf{4 \times 3}$ matrices of the linear systems, we have to add to the 12 equations in the original system, the 24 differential equations for the partial derivatives with respect to the parameters $\boldsymbol{v_x, v_y}$:

$$\frac{\partial x_i}{\partial v_x}(t), \frac{\partial y_i}{\partial v_x}(t), \frac{\partial vx_i}{\partial v_x}(t), \frac{\partial vy_i}{\partial v_x}(t), i = 1, 2, 3.$$

$$\frac{\partial x_i}{\partial v_y}(t), \frac{\partial y_i}{\partial v_y}(t), \frac{\partial vx_i}{\partial v_y}(t), \frac{\partial vy_i}{\partial v_y}(t), i = 1, 2, 3.$$

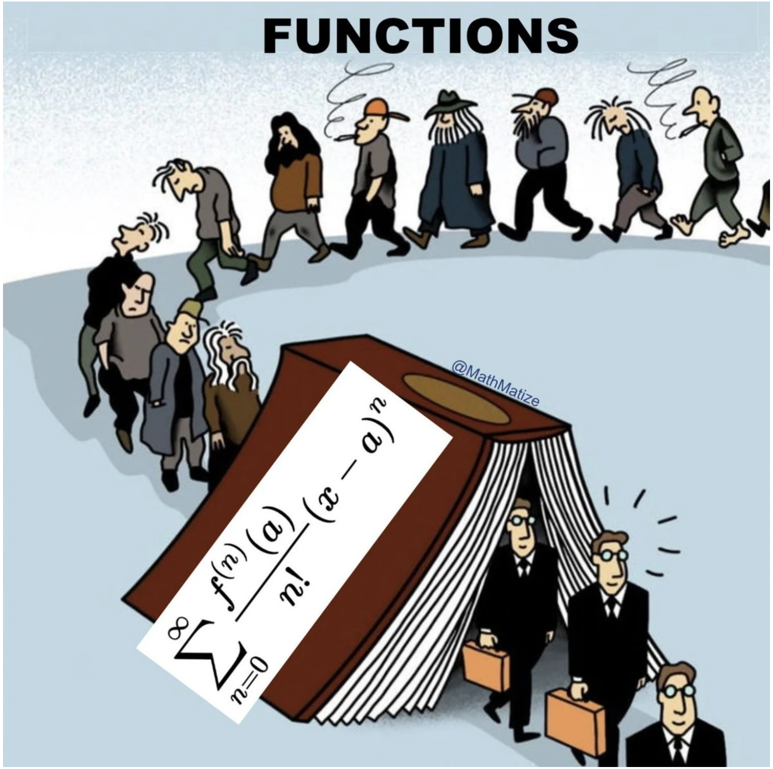
These equations can be obtained by differentiation of the original system with respect to the parameters $\boldsymbol{v_x, v_y}$, but we do not need them in explicit form for the ODE solver we will used.

A crucial decision for the success of finding periodic orbits is the choice of the numerical algorithm for solving this system of 36 ODEs. We use high order high precision Taylor Series Method (TSM).

Taylor Series Method (TSM)

Joseph-Louis Lagrange termed Taylor's theorem

"The main foundation of differential calculus."



Taylor Series Method (TSM)

For the initial value problem $\dot{u}(t) = f(u, t)$, $u_0 = u(0)$, the N-th order Taylor series method for finding an approximate solution $U(t) \approx u(t)$ is given by:

$$U(t+\tau) = \sum_{i=0}^{N} U^{[i]} \tau^i, \quad U^{[i]} = \frac{1}{i!} \frac{dU^{(i)}(t)}{dt^i},$$

where the coefficients $U^{[i]}$ are called normalized derivatives.

The use of an adaptive step-size strategy is crucial for the three-body problem. The time stepsize τ is determined this way:

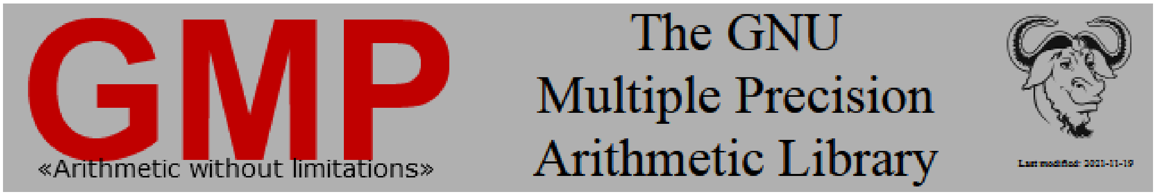
$$\tau = \frac{e^{-0.7/(N-1)}}{e^2} \min\left\{ \left(\frac{1}{\|\mathrm{U}^{[\mathrm{N}-1]}\|_\infty} \right)^{\frac{1}{N-1}}, \left(\frac{1}{\|\mathrm{U}^{[\mathrm{N}]}\|_\infty} \right)^{\frac{1}{N}} \right\}$$

Jorba, Angel, and Maorong Zou *Experimental Mathematics 14.1 (2005): 99-117*

Normalized derivatives are computed with the rules of automatic differentiation!

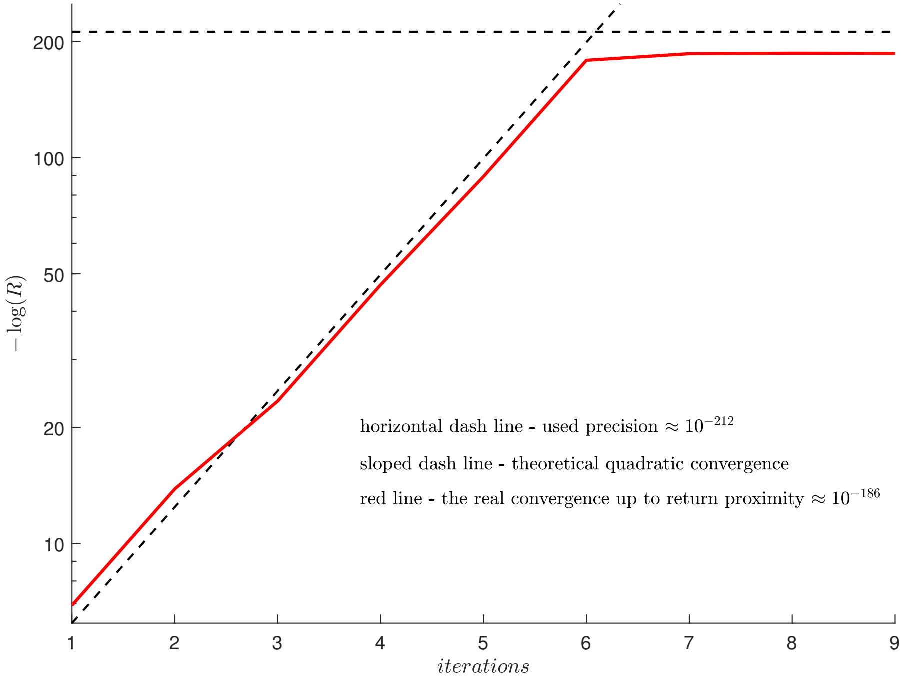
Parameters of simulations and accuracy analysis

- As numerical experiments show, order 22 per each 64 bit mantissa (bit of precision) is an optimal choice.
- At stage I precision of 128 bit ($\approx \mathbf{38.5}$ decimal digits) and 44 order TSM are used.
- At stage II precision of 192 bit ($\approx \mathbf{57.8}$ decimal digits) and 66 order TSM is used. A periodic solution is captured if $\boldsymbol{R_e(\overline{T}) < 10^{-20}}$. Each captured solution is additionally specified up to $\boldsymbol{R_e(\overline{T}) < 10^{-50}}$ by computations with increased precision of 320 bit ($\approx \mathbf{96.3}$ decimal digits) and 110 order TSM.
- At stage III we gradually increase the precision and Taylor series order starting with 448 bit of precision and 154 order. The stage may consist of several substages, depending on the accuracy we want. The iterations are until convergence (until return proximity stops to decrease).



Back
Close

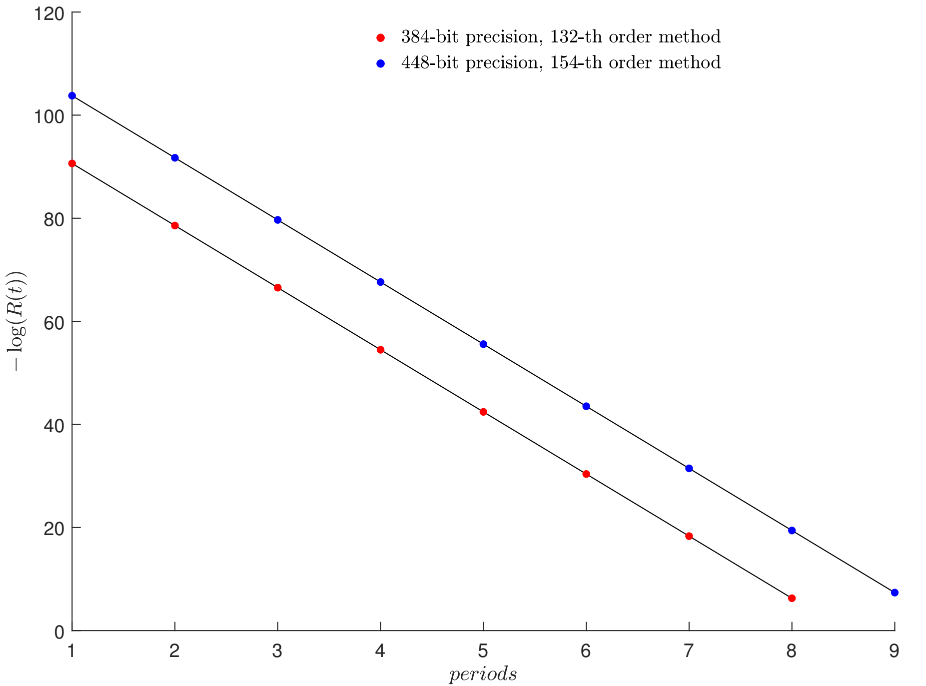
Convergence of Newton's method for "a difficult" i.c.



"A difficult" for computing i.c. means that this is the i.c. for which the minimal return proximity much below the used precision (for the example about 26 digits below the used precision). For this picture 704 bit of precision and 242 order TSM is used.

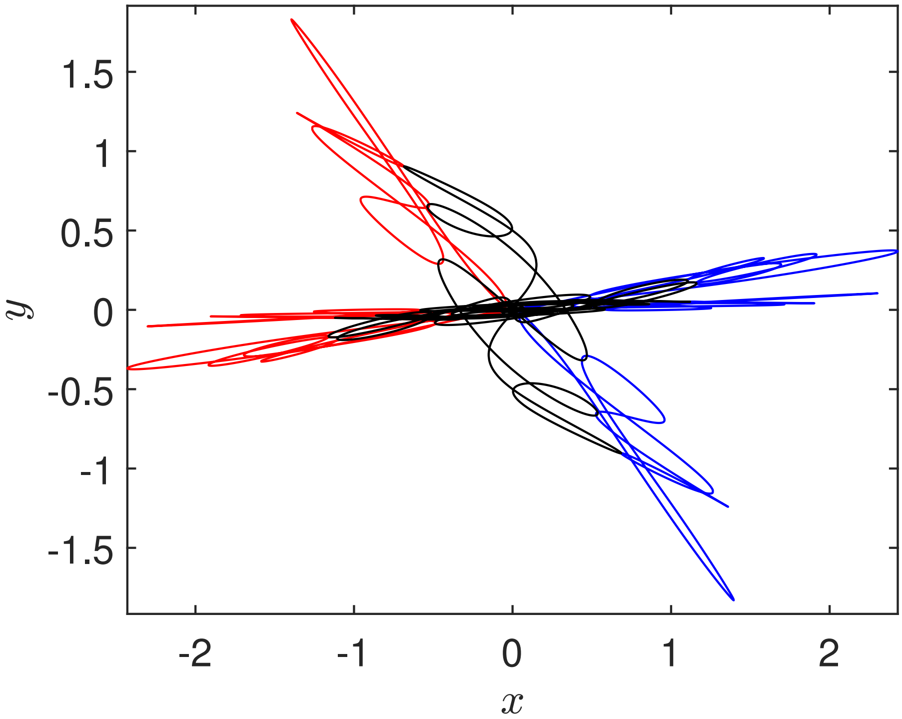
Back
Close

Return proximity $R(t)$ evolution of the "difficult" i.c. for i.c given with 116 decimal digits corresponding to 384 bit precision

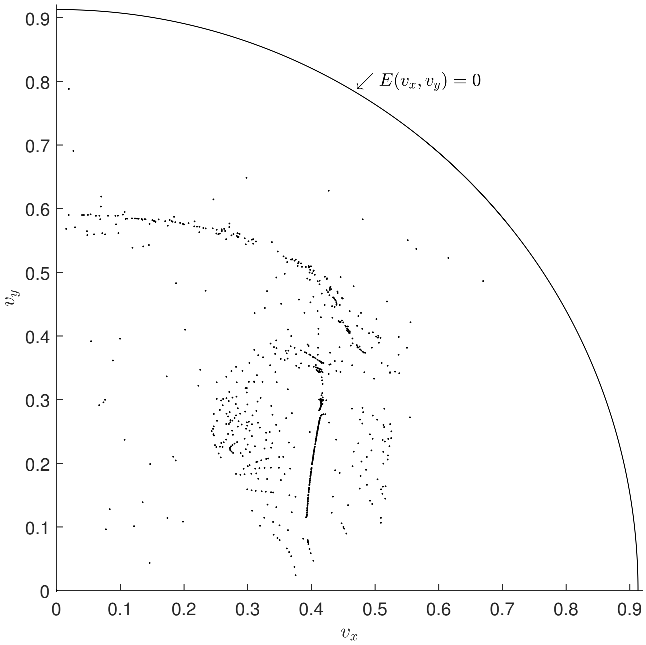


Back
Close

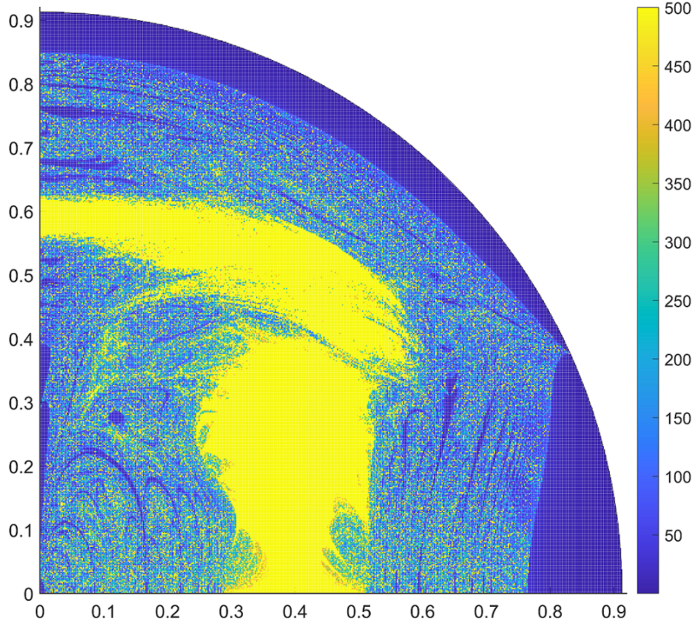
Real space plot for the “difficult” i.c.



Li and Liao's i.c.s and colormap of escape times

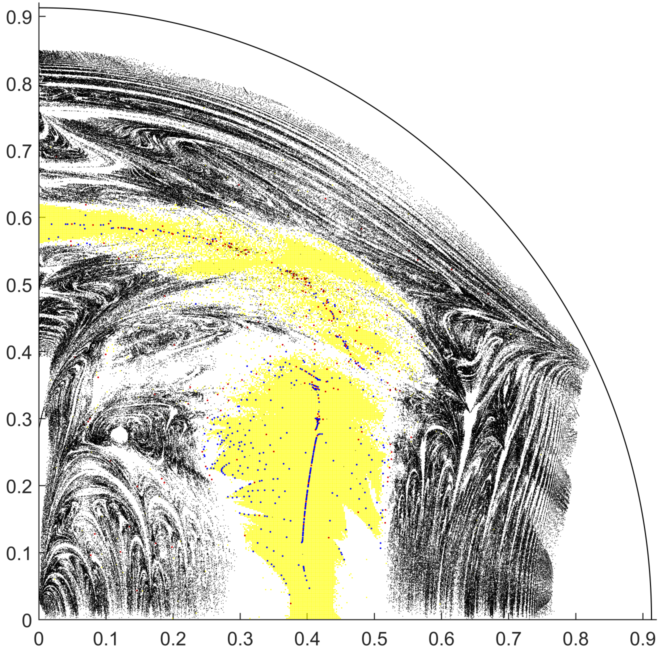


(Li and Liao's 695 i.c.s)

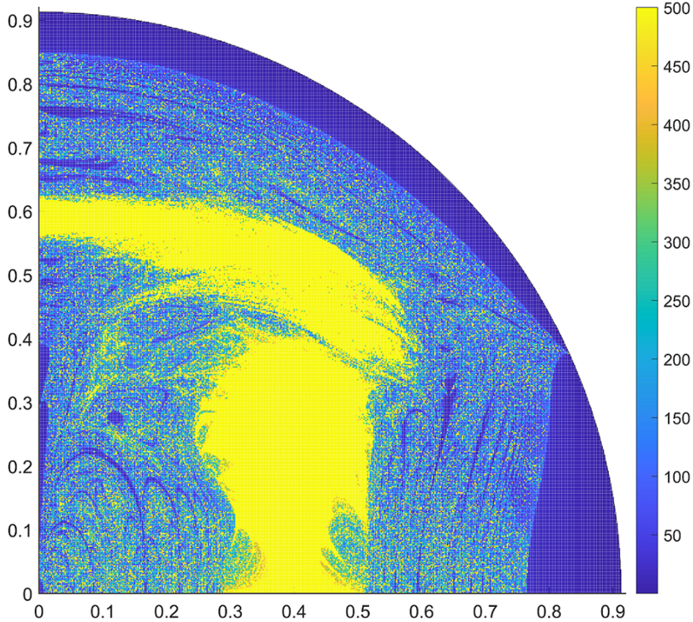


(Martynova et. al, *Astronomy reports 53 (2009)*)

The newly found Euler i.c.s and colormap of escape times

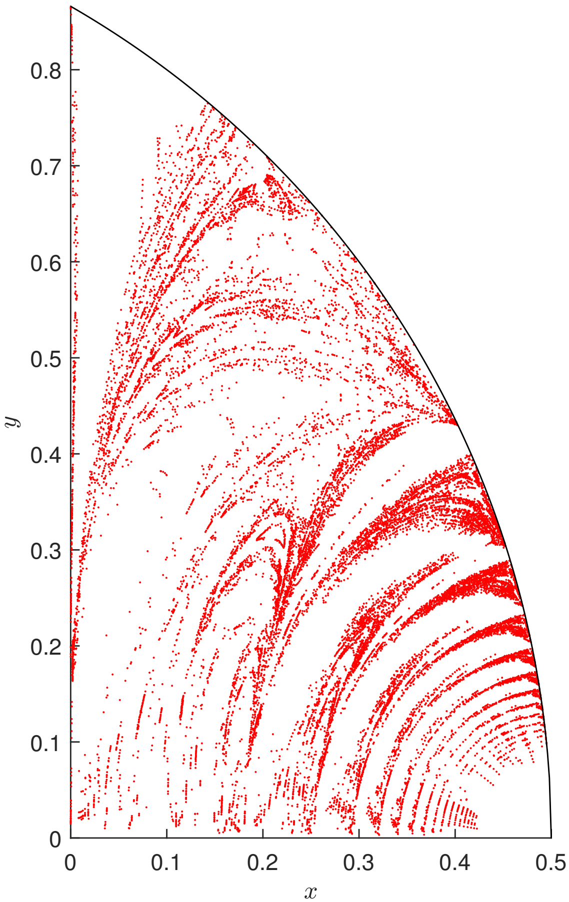


(More than 400,000 i.c.s)

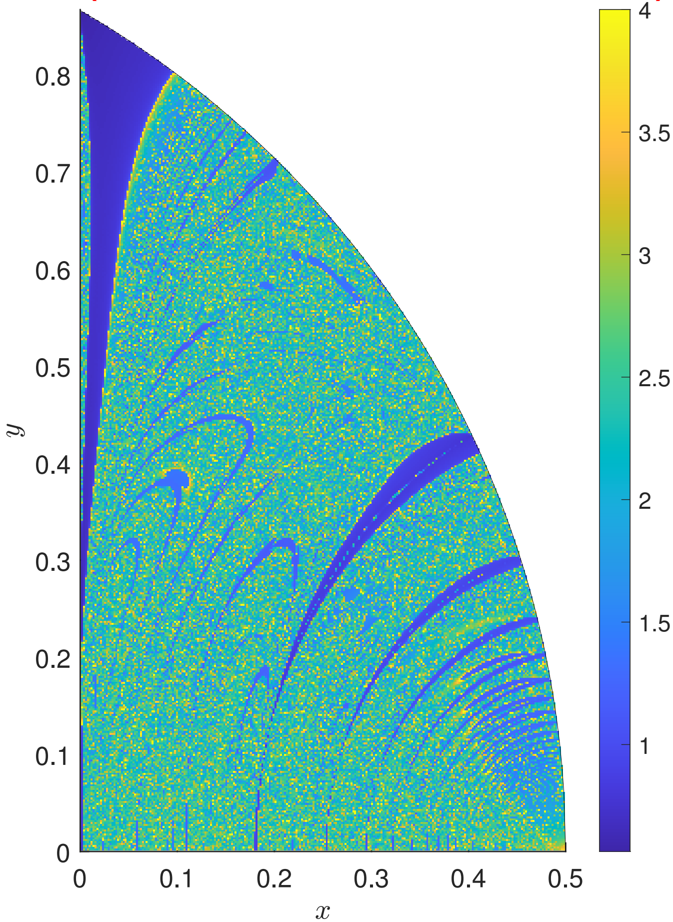


(Martynova et. al, *Astronomy reports 53 (2009)*)

FF results - 24,582 i.c.s found ("Maasai shield" structure)

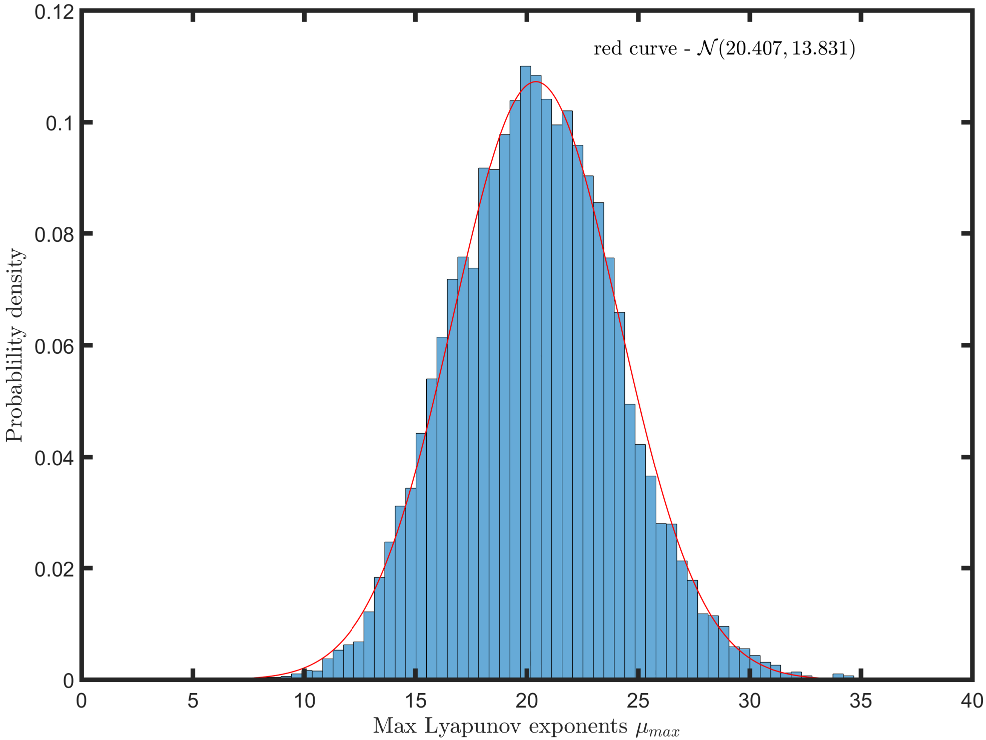


Distribution of i.c.s



escape times

Distribution of the maximal Lyapunov exponents of free-fall periodic orbits



The two faces of the three-body problem (Veljko's notes)

There is no doubt in my mind that this study will change the whole "atmosphere" around free-fall orbits, and that people will recognize the free-fall orbits as the second face of the three-body problem. This is somewhat like the Roman mythological god Janus who had two (identical) faces which could not be told apart, except here the two faces are different: one "face" of 3-b. problem is regular, with many stable orbits and the other is chaotic without a single stable orbit. It gives credence to the oft-repeated, but never justified mantra that the 3.b. problem is chaotic: well, it is if you look at it from one end, but not otherwise.

Recently published papers

Hristov, I., Hristova, R., Dmitrašinović, V., Tanikawa, K.
"Three-body periodic collisionless equal-mass free-fall orbits revisited"
Celestial Mechanics and Dynamical Astronomy, 136(1), 7, 2024

Hristov, I., Hristova, R., Dmitrašinović, V., Tanikawa, K.
"Instability of three-body periodic collisionless equal-mass free-fall orbits"
In Journal of Physics: Conference Series (Vol. 2910, No. 1, p. 012030). IOP Publishing, 2024

Hristov, I., Hristova, R. *"An efficient approach for searching three-body periodic orbits passing through Eulerian configuration"*
Astronomy and Computing, 49, 100880, 2024

Hristov, I., Hristova, R., Puzynin, I., Puzynina, T., Sharipov,Z., Tukhliev,Z.
"Searching for New Nontrivial Choreographies for the Planar Three-Body Problem" Physics of Particles and Nuclei, 55(3), 495-497, 2024

ACKNOWLEDGMENTS

We acknowledge the access to the Nestum cluster at HPC Laboratory, Research and Development and Innovation Consortium, Sofia Tech Park.

We are also grateful for the opportunity to use the resources of the HybriLIT platform, JINR, Dubna, Russia.

We thanks Richard Montgomery for the prompt reply of the questions concerning the “half period” property and for the proof of this property.

THANK YOU FOR YOUR ATTENTION!